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VIBRATIONS OF SANDWICH PLATES WITH CONCENTRATED MASSES AND SPRING-LIKE INCLUSIONS

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Linear dynamics of a sandwich beam (a plate of sandwich composition in one-dimensional cylindrical bending) bearing concentrated masses and supported by springs is described in the framework of the sixth order theory of multilayered plates. Analysis of the influence of a single inclusion and of a pair of identical inclusion upon vibrations of an infinitely long beam is performed by the use of the Green function method. To construct the Green functions, a dispersion polynomial is derived and normal modes are obtained. Parameters of propagating low-frequency waves are checked against results available in the literature. Then the Green functions for flexural and shear vibrations of a beam excited by a point force or a point shear moment are considered. Attention is focused on a comparison of forced vibrations of homogeneous beams and beams bearing concentrated masses supported by springs. The role of interaction of dominant flexural waves with dominant shear waves near inclusions is discussed. Conditions of localization of flexural waves at these inhomogeneous are explored in respect of excitation parameters and parameters of sandwich composition. Radiated acoustic power is computed in the case of a homogeneous beam and in a trapped mode case to illustrate the importance of the localization effect for structural acoustics.

1. INTRODUCTION

In the literature on linear dynamics of thin-walled structures, much attention has been paid to the interaction of flexural propagating waves with inhomogeneities such as concentrated masses or stiffeners [1, 2], and it has been shown that this interaction may generate intense vibrations near such an inclusion. This effect is analogous to the well-known phenomenon of mode trapping in acoustical waveguides, which has been thoroughly studied [3, 4]. However, in "elastic waveguides" like thin-walled plates and shells, similar effects are more complicated because wave propagation is governed by differential operators of order higher than those in acoustical waveguides, so that the interaction of waves of various types may be involved in the localization of vibrations in

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elastic waveguides. In the case of a simple Kirchhoff beam or plate model, a travelling flexural wave generates an evanescent wave at an inhomogeneity [5], while in reference [6] similar effects have been discussed for the case of T-joint beams (plates) when a pure longitudinal wave propagating in one member interacts with flexural waves in other members. In the case of Kirchhoff shell theory, the same phenomena of wave interaction has been analyzed in reference [7] for a cylindrical shell, with emphasis placed on the interaction of dominantly tangential and dominantly flexural waves.

In all the above-mentioned cases, inhomogeneities produce the effect of localized amplification of flexural vibrations (mode trapping), and in terms of structural acoustics this provides the possibility of intense sound radiation from this part of a structure. The classical formulation of the trapped mode problem is relevant to an interaction between inclusions and a travelling incident wave coming from infinity, but from the practical viewpoint it is probably more relevant to specify excitation conditions in terms of external loading. A case of most interest is then the forced vibrations of an infinitely long beam loaded by a concentrated force or moment and bearing concentrated inclusions. Then the Green function method [8] appears to be a convenient tool for solving this problem. This method is especially efficient when the Green function is available in a simple analytical form, as is the case for our sandwich plate theory.

The model of flat sandwich beam (or plate) is probably the simplest one which permits interaction between flexural and tangential waves similar to that observed in thin shells. In this case, dominantly tangential motion is associated with shear deformation generated by an in-plane "sliding" of the skins. This degree of freedom is independent of the lateral deflection, w, of the whole structure, and it is specified by an additional variable—the shear angle θ. At inhomogeneities such as concentrated masses or stiffeners, the shear wave produces conventional, predominantly bending, waves (similar to those existing in a Kirchhoff beam) and therefore amplitudes of lateral displacement increase.

There are many publications devoted to the derivation of theories of sandwich plates (see, for example, reference [9–14]) and to the analysis of their dynamical properties [15, 16]. It is not the goal of this paper to go into a detailed comparison of these theories, some of which are of rather high order and therefore predict the existence of a large number of various waves. However, as is well known, contributions to structural dynamics from the high order effects manifest themselves mostly at rather high frequencies: i.e., when a structural wavelength becomes of the same scale as the thickness of a core ply. In the present paper, attention is focused on the comparatively low-frequency range, with special reference to long travelling waves. Then governing equations of the sixth order [12, 13] can appropriately be adopted for a beam of sandwich composition.

Excitation of an infinitely long elastic waveguide bearing several concentrated masses or supported by several stiffeners may result in mode trapping between these inclusions; see a detailed analysis in reference [5]. (In particular, localization of flexural vibrations of membranes on an elastic foundation driven by transverse loading has been analyzed in reference [17].) In this paper, we show that the same phenomenon of trapping of flexural wave may occur for a sandwich beam both under transverse and shear excitation conditions.

2. THE SANDWICH BEAM MODEL

The theory of a sandwich beam is taken in the form suggested in reference [12]. An element of sandwich beam in its initial and deformed positions is shown in Figure 1(a). It consists of two symmetrical relatively thin, stiff skin plies and a thick, soft core ply.

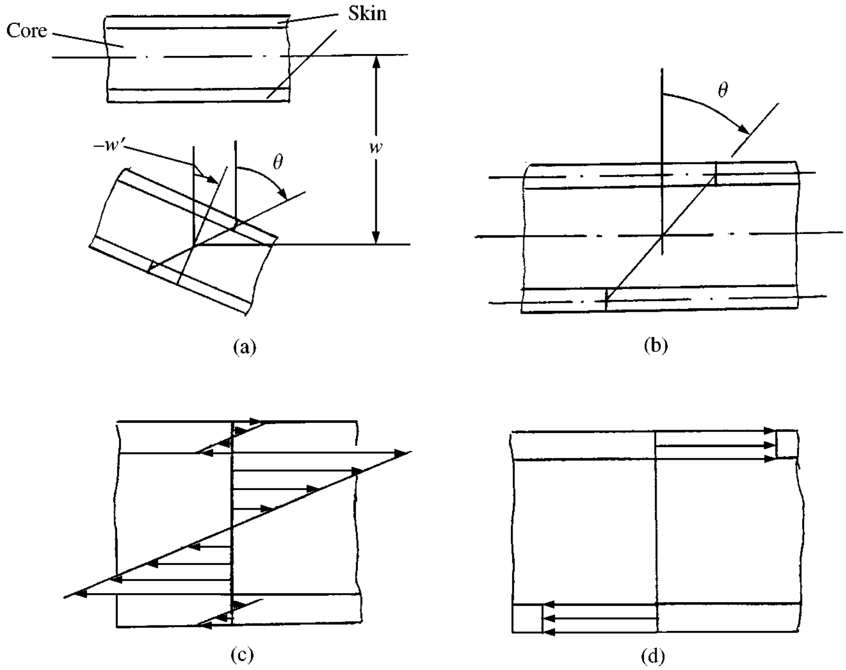

Figure 1. (a) An element of a sandwich beam in non-deformed and deformed positions: (b) shear deformation of a sandwich beam; (c) normal bending stresses in skin and core plies producing bending moment of the "first kind": (d) normal membrane stresses in skin plies producing bending moment of the "second kind".

Dimensionless parameters are introduced to describe the internal structure of the sandwich plate: $\varepsilon = h_{skin}/h_{core}$ as a thickness parameter (the ratio of the thickness of each individual skin ply to the thickness of the core ply), $\delta = \rho_{core}/\rho_{skin}$ as a density parameter, $\gamma = E_{core}/E_{skin}$ as a longitudinal stiffness parameter and $\gamma_g = G_{core}/G_{skin}$ as a shear stiffness parameter. Hereafter, subscripts denoting parameters of skin plies are omitted. The deformation of a sandwich beam element is governed by two independent variables: displacement of the mid-surface of the whole element w (which is the same for all plies), see Figure 1(a), and the shear angle between mid-surfaces of skin plies θ, see Figures 1(a) and 1(b) plotted for a case of pure shear deformation ($w = 0$).

The Hamiltonian of a vibrating composite beam (a plate undergoing cylindrical bending) in the absence of external forcing is

$$H = \frac{1}{2}\int_{t_1}^{t_2}\int_0^l [m(\dot{w})^2 + I_1(\dot{w}')^2 + I_2\dot{\theta}^2 - D_1\kappa_1^2 - \Gamma\tau^2]\,\mathrm{d}x\,\mathrm{d}t. \tag{1a}$$

Here the first term $m(\dot{w})^2$ is the sum of the kinetic energies of all plies in their vertical motion, while the second and the third terms correspond to rotation; the term $I_1(\dot{w}')^2$ is the sum of the rotational kinetic energies of all plies in their rotations about their own axes, and the term $I_2(\dot{\theta})^2$ is the rotational kinetic energy generated by sliding (tangential motion) of skin plies. The potential energy terms correspond to the energy of bending of each ply related to the curvature of the overall bending, the energy of membrane deformation in skin plies generated by shear angle and the energy of longitudinal shear deformation in a core ply respectively.

The elastic parameters in equation (1a) are [12]

$$D_1 = \frac{Eh^3}{12(1-v^2)}\left(2+\frac{\gamma}{\varepsilon^3}\right), \quad D_2 = \frac{Eh^3}{2(1-v^2)}\left(1+\frac{1}{\varepsilon}\right)^2, \quad \Gamma = \frac{Eh}{2(1+v)}\left(1+\frac{\gamma}{\varepsilon}\right)^2 \gamma_g \varepsilon,$$

$$m = \rho h\left(2+\frac{\delta}{\varepsilon}\right), \quad I_1 = \frac{\rho h^3}{12}\left(2+\frac{\delta}{\varepsilon^3}\right), \quad I_2 = \frac{\rho h^3}{2}\left(1+\frac{1}{\varepsilon}\right)^2. \tag{1b}$$

Here, $E = E_{skin}$, $h = h_{skin}$, and the Poisson ratio v is assumed to be the same for all plies. The moments and forces are related to lateral displacement and shear angle as [12]

$$M_1 = -D_1 w'', \qquad M_2 = D_2\theta', \qquad Q = \Gamma(\theta + w'). \tag{1c}$$

M_1 is a bending moment of the "first kind", composed as the sum of the bending moments acting in each ply, and produced by normal stresses linearly distributed in each ply in accordance with the classic Bernoulli–Euler model applied to each ply individually; see Figure 1(c). These stresses are related to the curvature $\kappa_1 = w''$, which is the same for all plies. M_2 is a bending moment of the "second kind", generated by the uniform part of normal stresses acting in skin plies, see Figure 1(d). These stresses are produced by $\kappa_2 = \theta'$. The shear force Q in the core ply is uniformly distributed, and is proportional to the shear angle in the core ply $\tau = \theta + w'$.

Stationarity conditions for the functional (1a) are formulated as equations of motion:

$$D_1 w^{IV} - \Gamma(\theta' + w'') + m\ddot{w} - I_1 \ddot{w}'' = q_w, \tag{2a}$$

$$-D_2\theta'' + \Gamma(\theta + w') + I_2\ddot{\theta} = q_\theta. \tag{2b}$$

Here q_w is the lateral intensity of distributed force and q_θ is the distributed intensity of shear moment.

The boundary conditions are obtained as non-integral terms in integration of the functional (1a) by parts. They are

$$D_1 w'' = 0 \quad \text{or} \quad w' = 0, \tag{3a}$$

$$I_1\ddot{w}' + \Gamma(\theta + w') - D_1 w''' = 0 \quad \text{or} \quad w = 0, \tag{3b}$$

$$D_2\theta' = 0 \quad \text{or} \quad \theta = 0. \tag{3c}$$

The first condition (3a) is related to the bending moment M_1, composed as the sum of the bending moments acting in each ply. Alternatively, the overall slope is zero. Condition (3b) formulates the absence of a transverse force at the edge of a beam. Its first term presents a contribution of rotational inertia ("Timoshenko"-type term), the second one account for shear interfacial stresses between plies and the last term corresponds to a standard Bernoulli–Euler transverse force. The alternative formulation of this boundary condition implies the absence of lateral displacement at the edge of a beam. The last set of boundary conditions (3c) is introduced by an independent variable of shear angle θ. Either a bending moment M_2, generated by the uniform part of normal stresses acting in skin plies, should be absent or the shear angle is equal to zero.

Typical values of material parameters for skin and core plies are summarized in Tables 1 and 2, after reference [18], chapter 1. The theory of bending of sandwich beams suggested in references [12, 13] and described here is a generalization of the classic Timoshenko

TABLE 1

Typical parameters of skin plies

	Density $\rho(\mathrm{kg/m^3})$	Young's modulus $E(\mathrm{GPa})$
Very stiff skin, Al alloy	2800	80
Stiff skin, 1-D carbon fibre/epoxy	1600	180 (longitudinal/10 (transverse)
Soft skin, 2-D glass weave/polyester	1600	12
Very soft skin, glass mats	1600	8

TABLE 2

Typical parameters of core ply

	Density $\rho(\mathrm{kg/m^3})$	Shear modulus $G(\mathrm{MPa})$
"Soft" core, PVC	80	31
"Hard" core, PVC	200	85

theory [19]. Thus, a vector of generalized displacements has three components (w, w', θ). In the case of static deformation, this theory results in equations of equilibrium identical to those given in reference [20, chapter 3].

3. DISPERSION POLYNOMIAL NORMAL WAVES AND GREEN'S MATRIX OF THE SANDWICH BEAM

For definiteness, we consider in some detail the specific case of an unloaded sandwich structure composed of isotropic individual layers, so that $\gamma = \gamma_g$ and equations (2a, b) become

$$\frac{Eh^3}{12(1-\nu^2)}\left(2+\frac{\gamma}{\varepsilon^3}\right)w'''' - \frac{Eh}{2(1-\nu)}\left(1+\frac{1}{\varepsilon}\right)^2 \varepsilon\gamma(\theta' + w'')$$

$$+ \rho h\left(2+\frac{\delta}{\varepsilon}\right)\ddot{w} - \frac{\rho h^3}{12}\left(2+\frac{\delta}{\varepsilon^3}\right)\ddot{w}'' = 0, \quad (4a)$$

$$-\frac{Eh^3}{2(1-\nu^2)}\theta'' + \frac{Eh}{2(1+\nu)}\varepsilon\gamma(\theta + w') + \frac{\rho h^3}{2}\ddot{\theta} = 0. \quad (4b)$$

To construct the Greens functions for vibrations of an infinitely long sandwich beam it is necessary to analyze the dispersion relation for waves propagating in an unbounded structure. The loading in equations (2) are set to zero, and the solution is taken to be of the form

$$w = A\exp(kx - \mathrm{i}\omega t), \qquad \theta = B\exp(kx - \mathrm{i}\omega t). \quad (5a, b)$$

One can now substitute the displacement vector (5) into differential equations (4) to obtain

$$\left[\frac{Eh^3(2+\gamma/\varepsilon^3)}{12(1-v^2)}k^4 - \frac{Eh\gamma\varepsilon(1+1/\varepsilon)^2k^2}{2(1+v)} + \frac{\rho\omega^2h^3}{12}\left(2+\frac{\delta}{\varepsilon^3}\right)k^2 - \rho h\omega^2\left(2+\frac{\delta}{\varepsilon}\right)\right]A$$

$$-\frac{\gamma\varepsilon(1+1/\varepsilon)^2k}{2(1+v)}B = 0,$$

$$\frac{Eh\gamma\varepsilon k}{2(1+v)}A + \left[-\frac{Eh^3k^2}{2(1-v^2)} + \frac{Eh\gamma\varepsilon}{2(1+v)} - \frac{\rho h^3\omega^2}{2}\right]B = 0, \tag{6a}$$

and setting the determinant of this system of algebraic equations to zero, one obtains for $\gamma \ll 1$ (i.e., for a realistic case when the Young's modulus of the skin ply is much larger than that of the core) the dispersion polynomial in the form

$$(kh)^6 - \frac{\gamma\varepsilon}{1+v}[1+3(1+1/\varepsilon)^2](kh)^4 - 6(2+\delta/\varepsilon)\left(\frac{\omega h}{c}\right)^2(kh)^2 - 6(2+\delta/\varepsilon)\left(\frac{\omega h}{c}\right)^4$$

$$+\frac{6(2+\delta/\varepsilon)}{1+v}\gamma\varepsilon\left(\frac{\omega h}{c}\right)^2 = 0, \tag{6b}$$

where $c = \sqrt{E/\rho(1-v^2)}$.

The results of analysis of the dispersion polynomial (6b) are compared with the solution of the two-dimensional problem given in reference [15] for a sandwich beam modelled as three elastic layers of different parameters glued to each other. In this latter paper, the theory of elasticity is used. Consequently, the dispersion curves are obtained for a broader class of waves existing in an infinitely long structure, including the ones relevant to symmetric (with respect to mid-surface) motions of faces. These waves are not encountered in the one-dimensional model [12] used in our present paper, as they are relevant to very high frequencies of excitation. In Figure 2, dispersion curves relevant to propagating waves

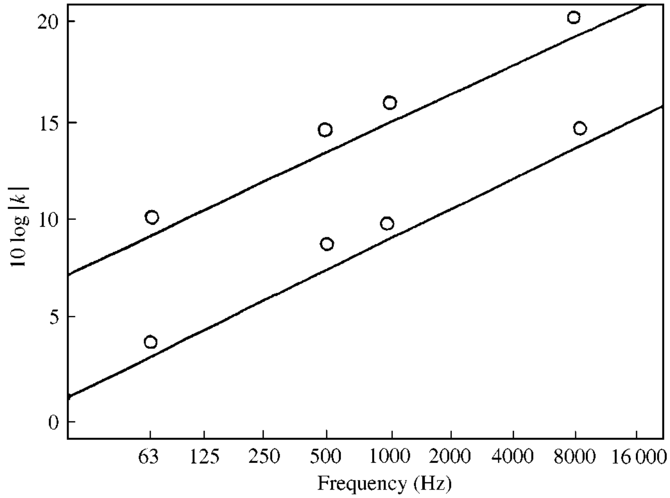

Figure 2. The dispersion curves of a sandwich beam. Solid lines after reference [15]; circles the present theory.

TABLE 3

Parameters of a laminated plate

	Thickness (mm)	E (N/m^2)	ρ (kg/m^3)	δ (%)	ν
Laminate	5	$1{\cdot}67 \times 10^{10}$	1760	2	0·3
Core	50	$0{\cdot}013 \times 10^{10}$	130	1·5	0·3

identified by both theories [15, 12] are presented for the set of parameters of a laminated plate shown in Table 3, specified in reference [15].

The upper line in Figure 2, reproduced from reference [15], corresponds to the wave number for dominantly shear waves propagating in a laminate, while the lower line gives the wave number for its pure bending. Only these kinds of motion are encountered in the model of sandwich beam suggested here and it is sufficient to describe waves of length larger than the thickness of a beam. Numerical analysis with the use of equation (6) gives wave numbers marked by circles in Figure 2. The agreement between results of calculations and data presented in reference [15] is fairly good. A detailed analysis of the dispersion polynomial for a sandwich beam is available in references [21, 22], so it is not reproduced here.

4. NORMAL WAVES AND GREEN'S MATRIX FOR SANDWICH BEAM

In the case of a sandwich beam, a normal mode has two components given by equations (4a, b), and the ratio of A_n and B_n, the coefficient of a normal mode, can be found for each root of the dispersion polynomial. This ratio may be obtained from any of two homogeneous algebraic equations derived from equations (5); for example from the first one we have

$$\beta_n = \frac{B_n}{A_n} = \frac{\gamma\varepsilon k_n}{2(1+\nu)}\left(1+\frac{1}{\varepsilon}\right)^2\left[\frac{h^2k_n^4}{6} - \frac{\gamma\varepsilon k_n^2}{2(1+\nu)}\left(1+\frac{1}{\varepsilon}\right)^2 - \left(\frac{\omega}{c}\right)^2\left(2+\frac{\delta}{\varepsilon}\right) + \frac{h^2k_n^2}{6}\left(\frac{\omega}{c}\right)^2\right]^{-1}. \tag{7}$$

Thus, for each resonant wave number k_n, a normal wave is defined up to flexural amplitude A_n that remains undetermined, while the shear amplitude B_n is given by equation (7). The dispersion polynomial is bi-cubic in k^2, and has three roots; it can easily be shown [21, 22] that there are travelling and evanescent dominantly flexural normal waves, while the third normal wave is dominated by shear displacements.

In the low-frequency range, the dominantly shear wave is of an evanescent type, but as the frequency exceeds a certain threshold value, it becomes a travelling wave. This is illustrated by the set of graphs in Figure 3(a), displaying the dependence of the wave number of the propagating shear wave on the frequency parameter for a set of values of the shear stiffness parameter γ. Thus, in Figure 3(b) the modal coefficient β is plotted versus frequency parameter for the same set of γ. A decrease in γ means that link between skin plies is getting weaker, resulting in the de-coupling of shear and bending motions of the sandwich beam, so that a shear wave is produced by simple parallel sliding of faces weakly connected by the core ply. A decrease in parameter γ makes the cut-off frequency smaller. Simultaneously, the wave number of the shear wave asymptotically tends to be a linear function of frequency

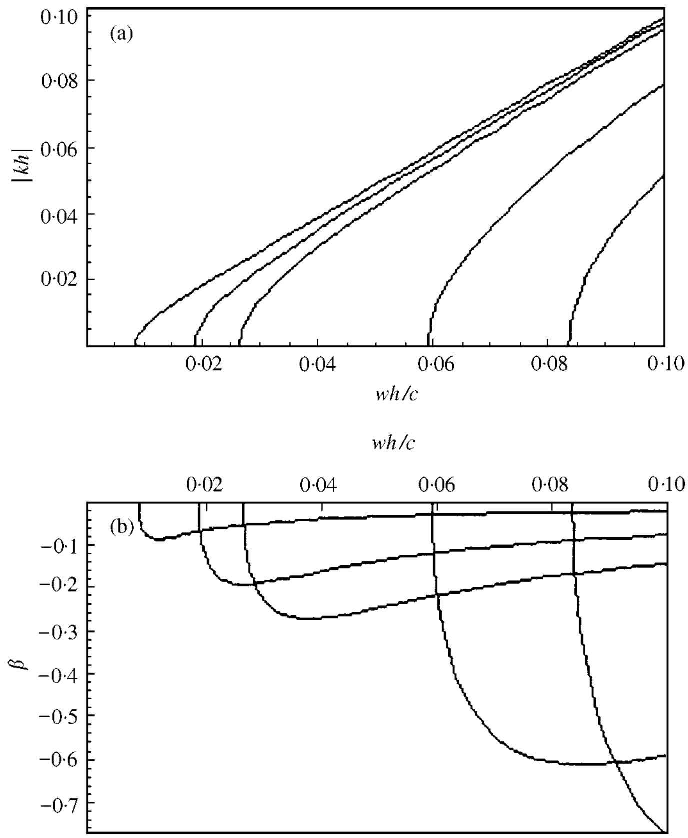

Figure 3. The influence of parameter γ on the shape of dispersion curves for propagating shear wave; values for γ from the left to the right: 0·001; 0·005; 0·01; 0·05; 0·1. (b) The influence of parameter γ on modal coefficient β for propagating shear wave; values for γ from the left to the right: 0·001; 0·005; 0·01; 0·05; 0·1.

because the reduced dispersion polynomial for in-plane sliding of the skins (with the coupling term in equation (4b) neglected) is of the second order. Consequently, the modal coefficient becomes small and weakly dependent on frequency.

To perform the analysis of forced vibrations of a sandwich beam, it is convenient to construct the Greens matrix for an infinitely long sandwich beam. This matrix is easy to derive since the roots of the dispersion polynomial and the modal coefficient are readily available. The elements of the Green matrix are composed of linear combinations of normal modes,

$$W_n(x,\xi)=\sum_{j=1}^{3}A_{nj}\exp(k_j|x-\xi|),\qquad \theta_n(x,\xi)=\sum_{j=1}^{3}A_{nj}\exp(k_j|x-\xi|),\tag{8a, b}$$

for three distinct loading conditions at the arbitrary point ξ:

(1) loading by a unit transverse force,

$$\Gamma\left[\theta_1(x,\xi)+\frac{\partial W_1(x,\xi)}{\partial x}\right]-D_1\frac{\partial^3 W_1(x,\xi)}{\partial x^3}-I_1\,\omega^2\frac{\partial W_1(x,\xi)}{\partial x}=\tfrac{1}{2}\operatorname{sign}(x-\xi),$$

$$\theta_1(x,\xi)=0,\qquad \partial W_1(x,\xi)/\partial x=0;\tag{9a}$$

(2) loading by a unit bending moment,

$$D_1 \frac{\partial^2 W_2(x,\xi)}{\partial x^2} = \tfrac{1}{2}\,\mathrm{sign}(x-\xi),\; W_2(x,\xi)=0, \qquad \partial\theta_2(x,\xi)/\partial x = 0; \tag{9b}$$

(3) loading by a unit shear moment,

$$D_2 \frac{\partial^2 \theta_3(x,\xi)}{\partial x} = \tfrac{1}{2}\,\mathrm{sign}(x-\xi),\; W_3(x,\xi)=0, \qquad \partial^2 W_3(x,\xi)/\partial x^2 = 0. \tag{9c}$$

The three roots of the dispersion polynomial (6b) which appear in equations (8a, b, 9a–c) are selected to satisfy the Sommerfeld radiation conditions at infinity, so that for $j=1,2,3$ $\mathrm{Re}(k_j)<0$ or, if $\mathrm{Re}(k_j)=0$, then $\mathrm{Im}(k_j)>0$. This latter choice is dictated by our selection of time dependence as $\exp(-\mathrm{i}\omega t)$, so as to ensure that the phase velocity of propagating waves is directed away from a source of excitation. We note that in the presence of mean flow in the fluid the connection between mode phase velocity and the radiation condition is significantly more complicated; see reference [23]. However, in the zero-flow situation considered here it is well known that the phase velocity can be used to determine the spatial location of each mode.

Substitution of equations (8) into each set of equations (9) gives three systems of linear algebraic equations in A_{nj}:

for the transverse loading case,

$$\sum_{j=1}^{3} [\Gamma(\beta_j + k_j) - D_1 k_j^3 - I_1\omega^2 k_j] A_{1j} = \frac{1}{2}, \quad \sum_{j=1}^{3} \beta_j A_{1j} = 0, \quad \sum_{j=1}^{3} k_j A_j = 0; \tag{9d}$$

for the bending moment case,

$$\sum_{j=1}^{3} D_1 k_j^2 A_{2j} = \frac{1}{2}, \quad \sum_{j=1}^{3} A_{2j} = 0, \quad \sum_{j=1}^{3} \beta_j k_j A_{2j} = 0, \tag{9e}$$

for the shear moment case,

$$\sum_{j=1}^{3} k_j^2 A_{3j} = 0, \quad \sum_{j=1}^{3} A_{3j} = 0, \quad \sum_{j=1}^{3} D_2 \beta_j k_j A_{2j} = \frac{1}{2}. \tag{9f}$$

The solution of these systems uniquely defines all the elements of the Green matrix of vibrations of a sandwich beam. The Green matrix is directly applicable to the analysis of vibrations of an infinitely long beam of sandwich composition, as it is formulated with the Sommerfeld conditions and loading conditions already taken into account.

5. VIBRATIONS OF A SANDWICH BEAM WITH A SINGLE INCLUSION

Consider an infinitely long sandwich beam bearing at the point $x = x_1$ a concentrated mass M supported by a linear spring of stiffness K; see Figure 4. Assume for simplicity that this inclusion does not produce inertial forces and moments in response to shear and rotational displacements and that the spring reacts only to vertical displacement. A driving generalized force (either a concentrated transverse force or a concentrated shear moment) of

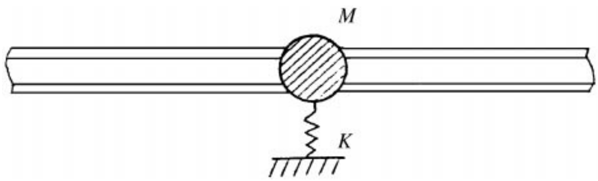

Figure 4. An inclusion in the sandwich beam.

unit amplitude and frequency ω acts at the point $x = x_0$. The Green functions relevant to loading conditions (9a) and (9c) formulate the shape of vibrations of a sandwich beam with no inclusion driven by a transverse force and by a shear moment respectively.

To account for the interaction between the beam and an inclusion it is necessary to formulate the equation of motion of the concentrated mass as

$$-M\omega^2 w_M = R - K w_M. \tag{10}$$

Here w_M is the displacement of the concentrated mass, R is the vertical force acting on the mass from the beam, and only stationary vibrations with circular frequency ω are considered.

Using the Green matrix, the amplitude of the displacements of the beam is given by applying the superposition principle

$$w(x) = F_{0n} W_n(x, x_0) - R W_1(x, x_1). \tag{11}$$

In equation (11), $F_{0n} = 1$ is a driving concentrated force of unit amplitude for $n = 1$ or a driving concentrated shear moment for $n = 3$ and it is equally applicable to both these excitation cases. Thus, functions W_n, $n = 1, 3$, are the components of the Green matrix formulating lateral displacement in response to transverse force or shear moment respectively. Note that the force R acts vertically downwards on the beam, leading to the minus sign in equation (11).

The continuity condition $w_M = w(x_1)$ permits us to find the reactive force R as

$$R = \frac{(K - M\omega^2) W_n(x_1, x_0)}{1 + (K - M\omega^2) W_1(x_1, x_1)} F_{0n}, \tag{12}$$

and the amplitude of forced vibrations of an infinitely long sandwich beam with inclusion is then given by the simple formula

$$w(x) = F_{0n}\left[W_n(x, x_0) - \frac{(K - M\omega^2) W_n(x_1, x_0)}{1 + (K - M\omega^2) W_1(x_1, x_1)} W_1(x, x_1) \right], \tag{13}$$

which in non-dimensional form becomes

$$\bar{w}(\bar{x}) = f_{0n}\left[\bar{W}_n(\bar{x}, \bar{x}_0) - \frac{\mu(\omega h/c)^2(\Omega^2/\omega^2 - 1)\, \bar{W}_n(\bar{x}, \bar{x}_1)}{1 + \mu(\omega h/c)^2(\Omega^2/\omega^2 - 1)\, \bar{W}_1(\bar{x}, \bar{x}_1)} \bar{W}_1(\bar{x}, \bar{x}_1) \right]. \tag{14}$$

Here $\Omega = \sqrt{K/M}$ is the eigenfrequency of an isolated mass supported by a spring, and $\mu = M/\rho h^3$ is a non-dimensional mass of the attachment. If $n = 1$, then a concentrated force is applied, and the non-dimensional parameter of a force is $f_{01} = F_{01}/Eh$. If $n = 3$, then

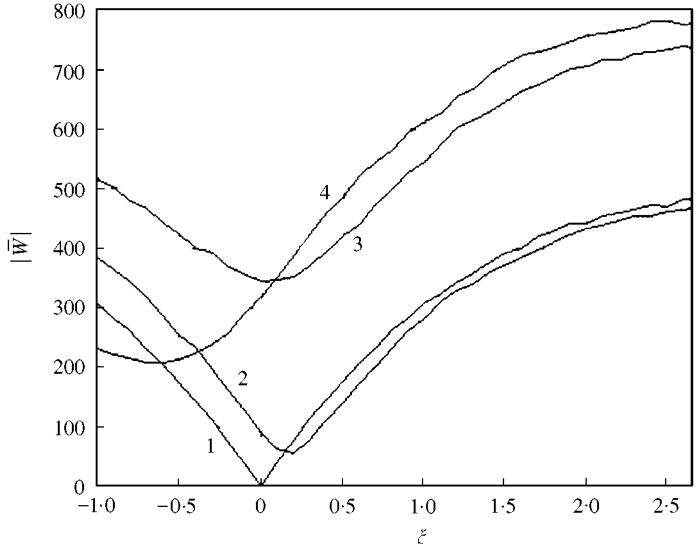

Figure 5. The influence of the mass parameter upon vibrations of a beam with a single inclusions.

a concentrated shear moment is applied and the non-dimensional parameter of a moment is $f_{03} = F_{03}/Eh^2$.

We now briefly consider several examples of vibrations of an infinitely long sandwich beam bearing a single concentrated inclusion, which are aimed just to demonstrate the effect of interaction of shear waves with an inclusion without carrying out a comprehensive parametric study. Therefore, sandwich plate composition parameters are selected as $\varepsilon = 0{\cdot}25, \delta = 0{\cdot}1, \gamma = 0{\cdot}001$ (this combination is typical of an "average" sandwich plate), and the influence of changes in values of these parameters upon vibrations of the beam with a single inclusion is not explored hereafter. Driving conditions are specified as excitation by a unit shear moment and no other driving force are considered. The driving shear moment is applied comparatively close to the inclusion at the distance $\bar{l} = l/h = 100$. It produces a propagating wave and the flexural displacement has both the real and the imaginary parts non-zero (time dependence is selected as $\exp(-\mathrm{i}\omega t)$). For brevity, we restrict our consideration to analysis of dependence of the modulus of the displacement on the inclusion's characteristics. In Figure 5, a set of curves presents absolute values of the non-dimensional lateral displacement $\bar{w} = |w|/h$ versus the non-dimensional axial co-ordinate $\xi = x/l$ scaled to the distance between a loading point and an inclusion. Thus, in Figure 5, a shear moment is applied at $\xi = 0$ and a concentrated mass is placed at $\xi = 1$. The stiffness of the spring is rather light, $\Omega/\omega = 10$, and the excitation frequency is fairly low, $\omega h/c = 0{\cdot}001$. Curve 1 is plotted for a sandwich beam with no attachment ($\mu = 0$). The loading conditions are given by equation (9c), so that there is no displacement at the excitation point. The discontinuity in the slope of curve 1, at $\xi = 0$, is explained by different signs of flexural displacements of the beam to the left and to the right of the shear moment. Curve 2 is plotted for $\mu = 1.$, and it is not much different from the previous one. Thus, we conclude that such a light inclusion does not distort wave propagation in the sandwich beam. However, as the mass increases up to $\mu = 5.$, the shape of vibrations is strongly influenced by this inclusion (see curve 3). Any further increase in inertia contributes less to the structural response and shape of vibrations since curve 3 at $\mu = 5.$, is closer to curve 4 at

$\mu = 100.$ than to curve 2 at $\mu = 1.$. This is clearly seen in Figure 5 for $x > 0$ and it could also be observed for $x < 0$ at somewhat larger distances from an excitation point.

6. VIBRATIONS OF A SANDWICH BEAM WITH TWO INCLUSIONS

The case considered in the previous section illustrates the influence of an isolated inclusion on forced vibrations of an infinitely long sandwich beam, but this is not associated with resonant behaviour typical of the trapped mode effect. This effect may easily be recognized in the case of excitation of a structure bearing two concentrated masses each supported by a spring. It manifests itself at certain resonant frequencies, and this will be investigated now.

Consider now two identical inclusions of mass M supported by linear springs of the same stiffness K placed at points x_1, x_2. The distance between these points is denoted as $l = x_2 - x_1$. Similar to the previous case, we assume that these inclusions respond only to vertical displacements of the beam. Either a driving vertical force or a shear moment of unit amplitude is applied at the point x_0. The Green function technique again permits one to obtain an elementary solution for such a problem. The equation of motion of each mass is formulated as

$$-M\omega^2 w_j = R_j - K w_j, \quad j = 1, 2, \tag{15}$$

while the amplitude of displacements of the beam at an arbitrary point is given by

$$w(x) = F_{0n} W_n(x, x_0) - R_1 W_1(x, x_1) - R_2 W_1(x, x_2) \tag{16}$$

(see discussion of cases $n = 1$ and $n = 3$ in the previous section).

Continuity conditions at $x = x_j, j = 1, 2$ give the following system of linear algebraic equation for the amplitudes of the displacements of the concentrated masses $m_j, j = 1, 2$:

$$\left[1 + \mu\left(\frac{\omega h}{c}\right)^2 \left(\frac{\Omega^2}{\omega^2} - 1\right) \bar{W}_1(\bar{x}_1, \bar{x}_1)\right] \bar{w}(\bar{x}_1) + \mu\left(\frac{\omega h}{c}\right)^2 \left(\frac{\Omega^2}{\omega^2} - 1\right) \bar{W}_1(\bar{x}_2, \bar{x}_1) \bar{w}(\bar{x}_2) = f_{0n} \bar{W}_n(\bar{x}_1, \bar{x}_0),$$

$$\mu\left(\frac{\omega h}{c}\right)^2 \left(\frac{\Omega^2}{\omega^2} - 1\right) \bar{W}_1(\bar{x}_1, \bar{x}_2) \bar{w}(\bar{x}_1) + \left[1 + \mu\left(\frac{\omega h}{c}\right)^2 \left(\frac{\Omega^2}{\omega^2} - 1\right) \bar{W}_1(\bar{x}_2, \bar{x}_2)\right] \bar{w}(\bar{x}_2) = f_{0n} \bar{W}_n(\bar{x}_2, \bar{x}_0). \tag{17}$$

Then the shape of forced vibrations of the sandwich beam becomes

$$\bar{w}(\bar{x}) = f_{0n}\left[\bar{W}_n(\bar{x}, \bar{x}_0) - \mu\left(\frac{\omega h}{c}\right)^2 \left(\frac{\Omega^2}{\omega^2} - 1\right) \bar{W}_n(\bar{x}_1, \bar{x}) \bar{w}(\bar{x}_1) - \mu\left(\frac{\omega h}{c}\right)^2 \left(\frac{\Omega^2}{\omega^2} - 1\right) \bar{W}_1(\bar{x}_2, \bar{x}) \bar{w}(\bar{x}_2)\right]. \tag{18}$$

It follows from equations (17) that a beam may perform vibrations between two masses when the determinant of this system of linear algebraic equations is zero, i.e.,

$$\left[1+\mu\left(\frac{\omega h}{c}\right)^2\left(\frac{\Omega^2}{\omega^2}-1\right)\sum_{j=1}^{3}A_{1j}\right]^2-\mu^2\left(\frac{\omega h}{c}\right)^4\left(\frac{\Omega^2}{\omega^2}-1\right)^2\left(\sum_{j=1}^{3}A_{1j}\exp(k_j l)\right)^2=0. \quad (19)$$

The resonant frequencies of these modes are detected by zeros of determinant (19). The wave numbers k_j, $j=1,2,3$, and the coefficients $A_j, j=1,2,3$, are frequency-dependent as specified by equations (6) and (9) so that this equation may be solved numerically. In particular, for the selected set of parameters $\varepsilon=0{\cdot}25$, $\delta=0{\cdot}1$, $\gamma=0{\cdot}001$, $l=l/h=100$, $\mu=10.$, $\Omega/\omega=1000.$ the resonant values of frequency parameter $\omega h/c$ are 0·00105 and 0·0024.

The trapped mode effect is easily demonstrated for an infinitely long sandwich beam loaded by a concentrated transverse force ($n=1$) at excitation frequency of $\omega h/c=0{\cdot}001$, i.e., very close to the first resonant frequency found from equation (19). A set of curves in Figure 6 presents an absolute value of flexural displacement in several excitation conditions. For convenience, in Figure 6 the axial co-ordinate is scaled to the distance between masses, $\xi=x/l$, so that the masses are positioned at the points $\xi=0$ and 1. A force is applied to the left of the span, at $\xi_0=-0{\cdot}5$ (curve 1), at the mid-span, $\xi_0=0{\cdot}5$ (curve 2) and to the right of the span, at $\xi_0=2{\cdot}5$ (curve 3). Curve 4 is plotted for the case of a concentrated transverse force acting at the point $\xi_0=0{\cdot}5$ of an infinitely long beam bearing no inclusions. In the selected part of a homogeneous structure, such a force produces a long flexural wave with a rather small curvature near the loading point. It is seen from these graphs that resonant vibrations of a beam between inclusions are generated regardless of the position of the force, but that the amplitude of vibrations is much larger when the force is applied at the centre of the beam. Similar to the previous figure, in Figure 6 an absolute value of amplitude is shown, so that discontinuities in slope at $\xi_1=0$, $\xi_2=1$ are explained by a change in sign of amplitude of vibrations between these points and outside the span.

We note that the existence of a trapped flexural mode excited by a transverse loading illustrated by Figure 6 is not specific to a sandwich beam, and may easily be found for

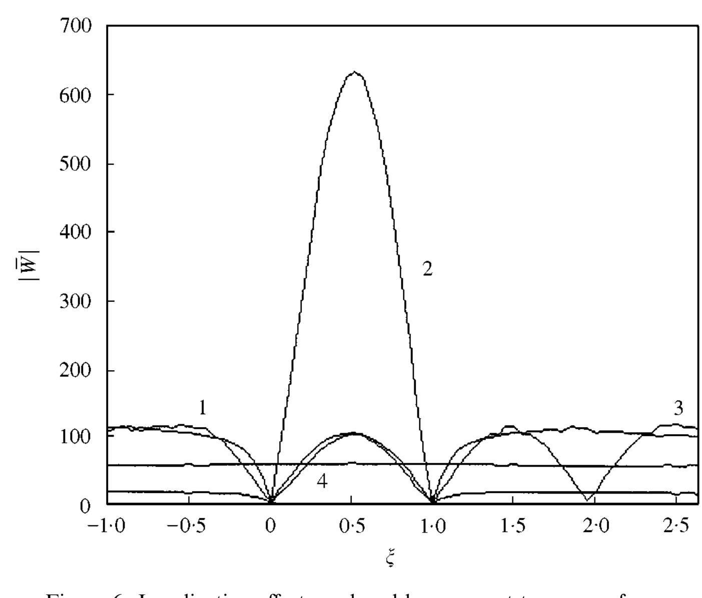

Figure 6. Localization effect produced by resonant transverse force.

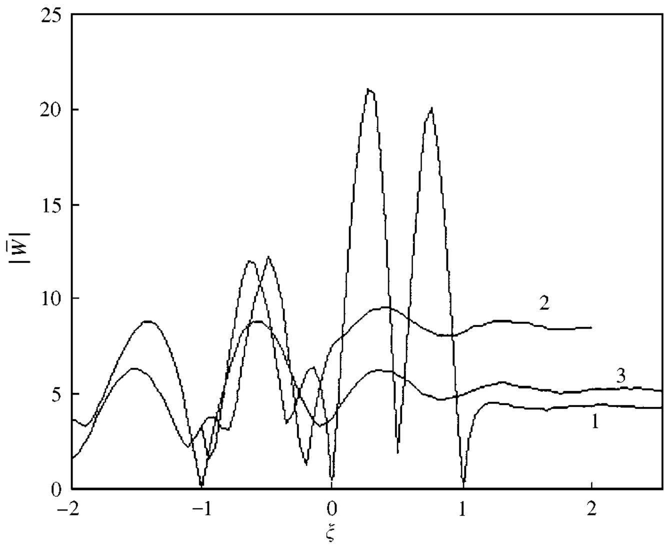

Figure 7. Localization effect in shear resonant excitation conditions.

a Kirchhoff beam model; see, for example reference [5]. However, for beams of sandwich composition, flexural modes may also be trapped in conditions of shear excitation. This effect is specific to their dynamics and is considered hereafter in more detail. In Figure 7, the shape of vibrations (the absolute value of displacement) of a beam having the same parameters is presented for loading by shear moment acting at $\xi = -1$, excitation frequency $\omega h/c = 0{\cdot}0024$. This is the second resonant frequency of flexural vibrations found from equation (19). Curve 1 is plotted for the case when the two masses are positioned at $\xi = 0$ and 1. The case of a single mass placed at $\xi = 0$ is illustrated by curve 2. Finally, curve 3 is plotted for a homogeneous beam. Although parameters of sandwich plate composition and parameters of inclusions are the same, there is a considerable difference in scales for flexural displacement between Figures 6 and 7, which is explained by the difference in excitation conditions. In Figure 6, a resonant transverse force acts at the mid-span of a beam, whereas in Figure 7, a resonant shear moment is applied outside the span between inclusions. The latter situation is of particular interest since it is relevant to trapping of an incident shear wave between inclusions and its transformation into the wave having significant flexural component in this zone. This case is most important from the acoustical viewpoint because intense radiation of sound may therefore be produced within the span. This aspect will be addressed in more detail in section 7. As follows from Figure 7, a single mass contributes only slightly to the shape of vibrations as compared with a homogeneous beam, whereas from curve 1 it is clear that vibrations are really trapped between two masses, in spite of the fact that the excitation moment acts out of the span.

The resonant trapping is strongly controlled by the parameters of the inclusions. This is illustrated by Figure 8 plotted for the case of resonant excitation of a beam by the shear moment acting at the centre of the span, $\xi_0 = 0{\cdot}5$. The parameters of a sandwich beam are the same as before, $\varepsilon = 0{\cdot}25$, $\delta = 0{\cdot}1$, $\gamma = 0{\cdot}001$, $\bar{l} = l/h = 100$. The inertial parameter of inclusion is $\mu = 10.$, which is held fixed. Curve 1 presents a shape of forced vibrations at $\Omega/\omega = 1000$, the excitation frequency is $\omega h/c = 0{\cdot}0024$ and resonant trapping is clearly seen. It should also be noticed that the amplitude of vibrations generated by a unit shear moment acting at the mid-span is several times bigger than in the case of the same shear moment

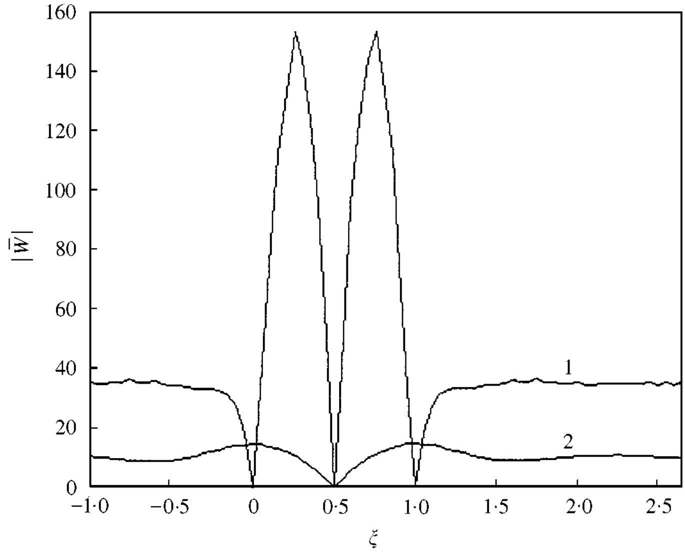

Figure 8. The influence of the stiffness parameter of inclusions on mode trapping.

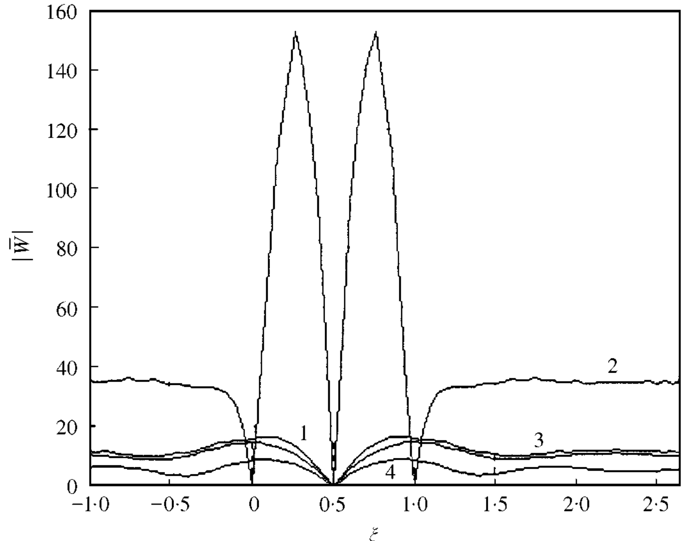

Figure 9. The influence of the stiffness parameter of a sandwich beam on mode trapping.

acting outside the span, see Figure 7. However, if the stiffness parameter of the inclusion is changed from $\Omega/\omega_{\cdot} = 1000$ to $\Omega/\omega_{\cdot} = 10$ (curve 2 in Figure 8) with all other parameters including excitation frequency unchanged, then the localization effect vanishes. Consequently, the shape of vibrations of the beam with two masses becomes the same as in the case of a homogeneous beam, the difference between curve 2 and the curve plotted for an homogeneous beam is not seen in Figure 8. The effect of localization of motion is also influenced by other parameters, specifically, by the stiffness parameter γ. In Figure 9, the set of parameters is $\varepsilon = 0{\cdot}25$, $\delta = 0{\cdot}1$, $\bar{l} = l/h = 100$, $\mu = 10.$, $\Omega/\omega = 1000.$, $\omega h/c = 0{\cdot}0024$. Curves 1 and 2 are plotted for $\gamma = 0{\cdot}001$; curves 3 and 4 are plotted for $\gamma = 0{\cdot}002$.

Curves 1 and 3 are quite close to each other, and represent the shape of vibrations of an infinitely long beam. Curves 2 and 4 are plotted for a beam bearing two masses, and it can clearly be seen that an increase in stiffness of the core material results in the elimination of the trapped mode effect. This is explained by a shift in resonant frequency of the trapped mode from a driven frequency of $\omega h/c = 0{\cdot}0024$ for a sandwich beam having $\gamma = 0{\cdot}002$.

7. RADIATED ACOUSTIC POWER FROM A SANDWICH BEAM IN RESONANT EXCITATION CONDITIONS

The trapped mode effect discussed in the previous section may result in a significant increase in radiated acoustic power as compared with the sound field from a homogeneous beam. To perform analysis of the acoustic field, it is necessary to formulate acoustical equations and their coupling with the equations of vibrations of the structure. In the present paper, the dynamics of the structure is described by integral equations, so that it is consistent and convenient to use the same approach for the acoustical part of the problem. Since an infinitely long flat plate in cylindrical bending is considered, the problem in acoustics is conveniently reduced to the Rayleigh integral [1], which takes into account continuity conditions at the fluid–structure interface and contains the free space Green function

$$p(x,0) = \frac{\mathrm{i}\rho_f \omega^2}{2} \int_{-\infty}^{\infty} \mathrm{H}_0^{(1)}\left(\frac{\omega}{c_f}|x-\xi|\right) w(\xi)\,\mathrm{d}\xi. \tag{20}$$

In equation (20), $p(x,0)$ is the contact acoustic pressure exerted at the fluid–structure interface and ρ_f, c_f are the fluid undisturbed velocity and the sound speed respectively.

In a coupled formulation of the problem in structural acoustics, the integral equations of motions of the structure, equation (11) or equation (16) with the contact acoustic pressure included along with a driving force, should be solved simultaneously with the integral equation of motion of the acoustic medium (20). Such a formulation is relevant to heavy fluid loading conditions [24], and as is well known, analyses of the acoustic field in a volume and of the structural response present serious difficulties even in the case of the simple model of a homogeneous Kirchhoff beam. However, the light fluid loading conditions (when the presence of an acoustic pressure is not essential in the formulation of structural dynamics) may be considered as a reasonable approximation for sufficiently stiff beams in contact with a relatively light medium. We explore here such a simplified formulation in order to estimate the possible increase in acoustic power generated by the trapped mode effect.

The power input from a concentrated force into an infinitely long plate is distributed between the energy transported to infinity through the structure in the form of travelling bending or/and shear waves and the energy emanating from the beam in the form of acoustic waves. The latter is formulated as [1]

$$N = \frac{1}{2}\,\mathrm{Re}\left\{\int_{-\infty}^{\infty} p(x,0)\,\bar{v}(x)\,\mathrm{d}x\right\} \tag{21a}$$

with $\bar{v}(x) = \omega\{\mathrm{Im}[w(x)] + \mathrm{i}\mathrm{Re}[w(x)]\}$ as a complex conjugate of the beam's velocity. For a homogeneous Kirchhoff beam, a detailed analysis of energy flows is available, for example, in reference [24].

In the present paper, our consideration is restricted to approximate estimation of the contribution made by the trapped mode effect to acoustic power radiated from a vibrating infinitely long sandwich beam. The results reported in the previous section show that outside the span between inclusions the shape of vibrations of a beam is not as different from that of a uniform beam as it is in the span between inclusions. Therefore, we compare acoustic power produced in the span of an infinitely long sandwich beam between inclusions with acoustic power radiated by the same zone of a homogeneous beam. Namely, we assume masses to be positioned at $x = 0, l$ and the driving force to be applied at $x = l/2$. Then we compute an acoustic power generated between points $x = 0, l$ with and without inclusions by the formula

$$N = \frac{1}{2}\mathrm{Re}\left\{\int_0^l p(x,0)\bar{v}(x)\,\mathrm{d}x\right\}. \tag{21b}$$

At the first step, a contact pressure is found by substitution of the amplitude of vibrations (11) or (16) into the Rayleigh integral (20). Then computation of the radiated power is performed straightforwardly by use of formula (21b). Or course, even in near-resonant conditions the effects of the trapped mode will extend beyond $0 \leqslant x \leqslant l$, so that a true measure of the effects on the acoustic power would, in principle, involve integration over all x in equation (21b). However, for the light fluid loading considered here we argue that these effects are small compared to the acoustic radiation generated by the large-amplitude motion in $0 \leqslant x \leqslant l$, justifying the finite integration range in equation (21b).

Numerical examples are presented for a beam of sandwich composition specified by elastic parameters given in reference [15] vibrating in water. In Figure 10, the dependence of non-dimensional acoustic power $\tilde{N} = N/\rho_f h c_f^3$ radiated from the segment $(0, l)$ of a beam upon a frequency parameter $\omega l/c$ is presented. Fluid loading parameters are $\rho_f/\rho = 0{\cdot}128$, $c_f/c = 0{\cdot}307$. Parameters of the sandwich beam composition are $\varepsilon = 0{\cdot}25$, $\delta = 0{\cdot}1$, $\gamma = 0{\cdot}001$, $\bar{l} = l/h = 100$; the inertial parameter of inclusions is $\mu = 10.$, and the excitation by a transverse concentrated force is considered. Curve 1 is plotted for a homogeneous beam (with no inclusions) and curves 2 and 3 are plotted for a beam with two inclusions having stiffness parameters of $\Omega/\omega = 10.$ and $\Omega/\omega = 1000.$ respectively. As is clearly seen, radiated

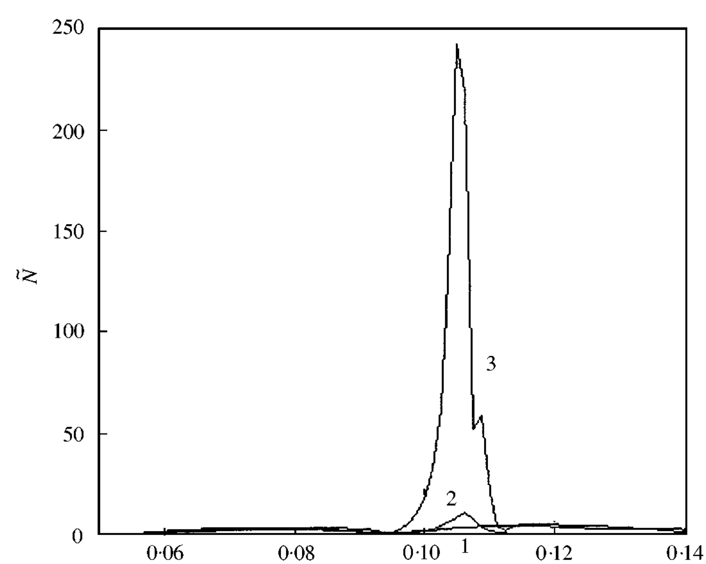

Figure 10. Non-dimensional radiated acoustic power versus excitation frequency in the case of unit transverse driving force acting at the middle of the span.

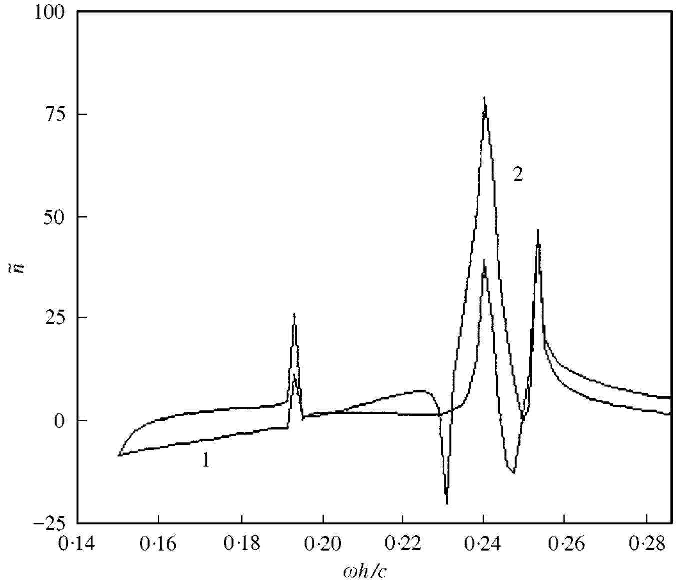

Figure 11. An acoustic power in dB (scaled by the power radiated from a homogeneous beam) versus excitation frequency in the case of unit shear moment acting at the middle of a span.

acoustic power strongly increase when inclusions become sufficiently stiff (curve 3), whereas weak supports (case 2) do not produce the mode trapping effect (as has also been observed in the previous section) and, hence, do not amplify sound radiation. This peak is reached at the resonant frequency of a trapped mode reported in the previous section. It is also remarkable that as the driving frequency deviates from resonance, the localization effect vanishes, and therefore the sound intensity becomes of the same level in all three cases.

In Figure 11, a dependence of radiated acoustic power upon frequency parameter $\omega l/c$ is presented in the vicinity of the second resonant frequency. The parameters of sandwich beam composition are the same as in the previous case. A concentrated shear moment is applied at the mid-span between inclusions. The radiated acoustic power is presented in dB with the reference level selected as power radiated from a beam with no attachments, $\tilde{n} = 10\log(N/N_0)$. Curve 1 is plotted for $\Omega/\omega = 10.$, curve 2 is plotted for $\Omega/\omega = 1000$. There is a substantial growth in intensity of sound radiation caused by the trapped mode effect around the resonant frequency detected by sending the determinant of equation (19) to zero. The resonant peak is much bigger in the case of a stiff inclusion than in the case of a relatively soft one. Outside the near-resonant excitation range there is not much difference between effects produced by the inclusion with different stiffness parameters.

8. CONCLUSIONS

The investigation into stationary vibrations of an infinitely long sandwich beam bearing concentrated masses supported by springs has been completed. The governing equations of motion have been derived by using Hamilton's principle, and a Green matrix for stationary vibrations of an infinitely long beam obtained analytically based on the analysis of the dispersion polynomial. Particular attention has been paid to the possibility of trapping flexural modes in excitation conditions relevant to the generation of dominantly shear waves in homogeneous structures. It has been shown that in the case of vibrations of a sandwich beam bearing two inclusions, intensive localized flexural vibrations may be provoked by a concentrated shear moment. Excitation of strong lateral vibrations occurs when a driving shear moment is applied between inclusions, but the same effect is also observed when a shear moment is acting outside the zone of intensive vibrations. The role of localization of motion is also explored from the viewpoint of sound radiation intensity. It is shown that a significant increase in acoustic power radiated from the span of a beam between inclusions occurs in the mode trapping case.

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